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Darboux transformations and the discrete KP equation

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Abstract. This paper presents two results. First it is shown how the discrete KP equation arises from a superposition principle associated with the Darboux transformation of the two-dimensional Toda system. Then Darboux transformations and binary Darboux transformations are derived for the discrete KP equation and it is shown how these may be used to construct exact solutions.

1. Introduction

In 1981, Hirota [2] introduced a discrete system which has since become one of the most widely studied fully discrete integrable systems in three dimensions. This was originally called a *discrete analogue of the generalized Toda equation* (DAGTE) because it was shown [2] that in one continuum limit it becomes the two-dimensional (A_∞) Toda system [7]. It was later shown [8] that this discrete system is the base member in a hierarchy which is equivalent, after a change of coordinates, to the KP hierarchy and for this reason it has also been called the *discrete KP equation* (dKP), a name we will adopt here.

There are two main aims of this paper. First it will be shown how the dKP equation and associated linear problem may be derived by considering Darboux transformations [6] for the two-dimensional Toda lattice. This work exactly follows the approach developed recently [9] in obtaining the discrete BKP equation (dBKP) from consideration of the Moutard transformation.

Second, as a by-product of this derivation, we obtain Darboux transformations applicable to the dKP equation. It is then shown that we may construct binary Darboux transformations in an exactly similar way to the continuous case. Using the basic and the binary Darboux transformations, classes of solutions of the dKP equation are obtained which generalize some solutions obtained by Ohta *et al* [11] using a direct approach.

2. The discrete KP equation and the two-dimensional Toda lattice

The DAGTE or dKP equation was introduced by Hirota [2] in the following form. Consider a function $F = F(m_1, m_2, m_3)$, and arbitrary constants Z_1, Z_2, Z_3 satisfying $Z_1 + Z_2 + Z_3 = 0$, then the dKP equation is

$$(Z_1 e^{D_{m_1}} + Z_2 e^{D_{m_2}} + Z_3 e^{D_{m_3}}) F \cdot F = 0 \tag{2.1}$$

where D_{m_i} are Hirota derivatives, or, written more explicitly,

$$Z_1 F(m_1+1, m_2, m_3) F(m_1-1, m_2, m_3) + Z_2 F(m_1, m_2+1, m_3) F(m_1, m_2-1, m_3) \\ + Z_3 F(m_1, m_2, m_3+1) F(m_1, m_2, m_3+1) = 0. \tag{2.2}$$

Introducing the change of independent variables

$$n_1 = \frac{-m_1 + m_2 + m_3}{2} \qquad n_2 = \frac{m_1 - m_2 + m_3}{2} \qquad n_3 = \frac{m_1 + m_2 - m_3}{2} \tag{2.3}$$

into (2.2) and writing $F(m_1, m_2, m_3) = \tau(n_1 - \frac{1}{2}, n_2 - \frac{1}{2}, n_3 - \frac{1}{2})$ then (2.2) becomes

$$Z_1\tau(n_1, n_2 + 1, n_3 + 1)\tau(n_1 + 1, n_2, n_3) + Z_2\tau(n_1 + 1, n_2, n_3 + 1)\tau(n_1, n_2 + 1, n_3) + Z_3\tau(n_1 + 1, n_2 + 1, n_3)\tau(n_1, n_2, n_3 + 1) = 0.$$

In [8, 11] for example, the parameters Z_i are given canonical values in terms of other parameters a_1, a_2, a_3 and the equation takes the form

$$(a_2 - a_3)\tau(n_1, n_2 + 1, n_3 + 1)\tau(n_1 + 1, n_2, n_3) + (a_3 - a_1)\tau(n_1 + 1, n_2, n_3 + 1)\tau(n_1, n_2 + 1, n_3) + (a_1 - a_2)\tau(n_1 + 1, n_2 + 1, n_3)\tau(n_1, n_2, n_3 + 1) = 0. \tag{2.4}$$

One may rescale the equation by means of the transformation

$$\tau \to Z_1^{-n_2n_3}(-Z_2)^{-n_1n_3}Z_3^{-n_1n_2}\tau \tag{2.5}$$

to remove the parameters Z_i and we get

$$\tau_1\tau_{23} - \tau_2\tau_{13} + \tau_3\tau_{12} = 0 \tag{2.6}$$

where here and below we use the notation $X_i = X|_{n_i \to n_i+1}$ ('increment n_i') so that, for example

$$\tau_1 := \tau(n_1 + 1, n_2, n_3)$$
$$\tau_{13} := \tau(n_1 + 1, n_2, n_3 + 1).$$

We will refer to (2.6) as the dKP equation. Note that by using a rescaling of the form (2.5) we may give the coefficients of the three terms in the dKP equation any value we wish.

The reason we choose to make the second term have coefficient -1 is to allow a succinct presentation of the associated linear problem [1] which for (2.6) takes the form

$$\phi_{ij} = \frac{\tau_i\tau_j}{\tau\tau_{ij}}(\phi_j - \phi_i) \qquad (1 \leqslant i < j \leqslant 3). \tag{2.7}$$

It may be readily shown that (2.7) are compatible in the sense that $(\phi_{12})_3 = (\phi_{13})_2 = (\phi_{23})_1$ if and only if τ satisfies (2.6).

Next, we recall some results for the two-dimensional Toda lattice. Let $z(n) = z(x, t, n)$ be a function of three variables, one discrete (n) and two continuous (x, t). The system

$$z_{xt}(n) - \mathrm{e}^{-z(n-1)} + 2\mathrm{e}^{-z(n)} - \mathrm{e}^{-z(n+1)} = 0 \tag{2.8}$$

where $n \in \mathbb{Z}$ was introduced by Mikhailov [7] and is known as the *two-dimensional Toda lattice*. More generally, there is a system of the form

$$z_{xt}(n) + \sum_m C^{nm}\mathrm{e}^{-z(m)} = 0 \tag{2.9}$$

which is known to be integrable when C is the Cartan matrix of any semi-simple or affine Lie algebra. See in particular [3, 4]. For this reason (2.8) is called, more specifically, the A_∞ Toda lattice.

The A_∞ Toda lattice has Lax pair

$$\begin{aligned} \phi_x(n) &= v(n)\phi(n) + \phi(n-1) \\ \phi_t(n) &= u(n)\phi(n+1) \end{aligned} \tag{2.10}$$

in which $v(n+1) - v(n) = z_x(n)$ and $u(n) = \mathrm{e}^{-z(n)}$. The coefficients $v(n)$ and $u(n)$ may be consistently parametrized in terms of $\tau(n) = \tau(x, t, n)$ as

$$v(n) = \left(\log \frac{\tau(n-1)}{\tau(n)}\right)_x \qquad u(n) = \frac{\tau(n+1)\tau(n-1)}{\tau(n)^2}. \tag{2.11}$$

Darboux transformations for (2.10) were found by Matveev [5, 6] and the reductions of this to more general Toda lattices have been studied recently [10]. The basic Darboux transformation is expressed in the following result.

Proposition 2.1. Given a non-zero solution $\theta(n)$ of (2.10),

$$\mathrm{DT}^{\theta}\colon \phi(n) \to \phi(n-1) - \frac{\theta(n-1)}{\theta(n)}\phi(n) \qquad \tau(n) \to \theta(n)\tau(n) \tag{2.12}$$

leaves (2.10) invariant.

We wish to use this Darboux transformation to introduce two discrete variables n_1, n_2. Roughly speaking, we think of a Darboux transformation DT^{θ^i} as giving rise to a change, actually in this case a *decrement*, in the discrete variable n_i. To be able to use the more convenient notation introduced above we relabel the existing discrete variable n as n_3. We will also use a modification of the subscript notation for increments to denote the corresponding decrements: $X_{i'} = X|_{n_i \to n_i - 1}$.

So now suppose that we have two eigenfunctions θ^1, θ^2 of (2.10) then the Darboux transformation (2.12) gives transformed quantities

$$\phi^i = \phi_{3'} - \frac{\theta^i_{3'}}{\theta^i}\phi \tag{2.13}$$

$$\tau^i = \theta^i \tau \tag{2.14}$$

for $i = 1, 2$. Further, after two Darboux transformations, determined by θ^1 and then by $\mathrm{DT}^{\theta^1}\,\theta^2 = \theta^2_{3'} - (\theta^1_{3'}/\theta^1)\theta^2$, we get

$$\tau^{12} = (\theta^1\theta^2_{3'} - \theta^1_{3'}\theta^2)\tau. \tag{2.15}$$

(2.15) using (2.14) gives

$$\tau^{12}\tau_{3'} = \tau^1\tau^2_{3'} - \tau^1_{3'}\tau^2 \tag{2.16}$$

which is a nonlinear superposition principle for solutions of (2.8) via the change of variables (2.11). Similarly, (2.13) gives

$$\phi^i = \phi_{3'} - \frac{\tau^i_{3'}\tau}{\tau_{3'}\tau^i}\phi \qquad (i = 1, 2). \tag{2.17}$$

Remark. Observe that the nonlinear superposition formula (2.16) is not invariant under the interchange of superscripts 1 and 2, in fact $\tau^{12} = -\tau^{21}$, and hence does not represent a permutability theorem in the usual sense. However, from (2.11) it is clear that this change of sign is irrelevant to the solutions of (2.8) and so we may think of (2.16) as a permutability theorem. This aspect has been discussed in more detail in [9] where a very similar situation exists. In that paper, there is also a careful discussion of the way in which we may legitimately convert the labels (superscripts) with respect to which τ is skew-symmetric into shifts on a lattice (subscripts) with respect to which τ is necessarily symmetric.

It is now clear that if we wish to regard the upper indices as shifts on a lattice and hence obtain (2.6), then these shifts should be interpreted as *decrements*. If we do this then we have

$$\tau_{1'2'}\tau_{3'} = \tau_{1'}\tau_{2'3'} - \tau_{1'3'}\tau_{2'}$$

which after the shift $n_i \to n_i + 1$ for $i = 1, 2, 3$ and a rescaling to the form (2.5), $\tau \to (-1)^{n_1 n_2}$, to change the sign of one term, gives (2.6). The same interpretation and transformation of (2.17) gives

$$\phi_{i3} = \frac{\tau_i \tau_3}{\tau \tau_{i3}}(\phi_3 - \phi_i) \qquad (i = 1, 2) \tag{2.18}$$

which constitutes two of the three equations in the Date *et al* [1] linear problem (2.7).

Remark.

(1) Since $\mathrm{DT}^{\theta_1}\, \theta_1 = 0$, and a nonzero solution is needed for the definition of a Darboux transformation we cannot obtain, for example, τ^{11}. Hence the interpretation of the action of the Darboux transformation as a shift on a lattice only works in a local sense. That is, it may be used to define the relationship of the values of the field τ at nearest-neighbour sites only.

(2) If we had instead started with the adjoint linear problem and its Darboux transformation, we would find that the action of the adjoint Darboux transformation may be interpreted as an increment.

3. Darboux transformations

We will now derive Darboux transformations for the the linear system

$$\phi_{ij} = \frac{\tau_i \tau_j}{\tau \tau_{ij}}(\phi_j - \phi_i) \qquad (1 \leqslant i < j \leqslant 3) \tag{3.1}$$

which are compatible $((\phi_{12})_3 = (\phi_{13})_2 = (\phi_{23})_1)$ if and only if

$$\tau_1 \tau_{23} - \tau_2 \tau_{13} + \tau_3 \tau_{23} = 0. \tag{3.2}$$

It follows from (3.1) that

$$\phi_{i'} - \phi_{j'} = \frac{\tau_{i'j'}\tau}{\tau_{i'}\tau_{j'}}\phi. \tag{3.3}$$

This form of the system is algebraically compatible (i.e. the three linear equations for unknowns $\phi_{1'}, \phi_{2'}, \phi_{3'}$ has a solution) if and only the same condition (3.2) holds. Also, from this form of the system we see that, for any two solutions θ, ϕ,

$$\theta\phi_{1'} - \theta_{1'}\phi = \theta\phi_{2'} - \theta_{2'}\phi = \theta\phi_{3'} - \theta_{3'}\phi \tag{3.4}$$

so that we may unambiguously define

$$C'(\theta, \phi) = \theta_{k'}\phi - \theta\phi_{k'} \tag{3.5}$$

where $k = 1, 2$ or 3.

We may now obtain the Darboux transformation.

Proposition 3.1. Given any non-zero solution θ of (3.1);

$$\mathrm{DT}^{\theta}\colon \phi \to \frac{C'(\theta, \phi)}{\theta} \qquad \tau \to \theta\tau \tag{3.6}$$

leaves (3.1) invariant.

As usual, we may obtain closed form expressions for the result of N applications of the above Darboux transformation. To do this we need to define the Casoratian of N solutions. Let $\boldsymbol{\theta} = (\theta^1, \ldots, \theta^N)^t$ be an N-vector solution of (3.1). From (3.3) it may be shown that, the Casorati determinant (with back-shifts)

$$C'(\theta^1, \ldots, \theta^N) = |\boldsymbol{\theta}, \boldsymbol{\theta}_{i'}, \ldots, \boldsymbol{\theta}_{\underbrace{i' \ldots i'}_{N-1}}| \qquad (1 \leqslant i \leqslant 3) \tag{3.7}$$

may also be unambiguously defined and so we use the notation

$$C'(\theta^1, \ldots, \theta^N) = |\boldsymbol{\theta}(0), \boldsymbol{\theta}(-1), \ldots, \boldsymbol{\theta}(1-N)| \tag{3.8}$$

where $\boldsymbol{\theta}(n)$ denotes the N-vector $(\theta^1, \ldots, \theta^N)^t$ subject to the shift $n_i \to n_i + n$, where $i = 1, 2$ or 3, the same value being taken for i in each column in the determinant. Then we have the following.

Proposition 3.2. Given N solutions $\theta^1, \ldots, \theta^N$ of (3.1) such that $C'(\theta^1, \ldots, \theta^N) \neq 0$,

$$\phi \to \frac{C'(\theta^1, \ldots, \theta^N, \phi)}{C'(\theta^1, \ldots, \theta^N)} \qquad \tau \to C'(\theta^1, \ldots, \theta^N)\tau \tag{3.9}$$

leaves (3.1) invariant.

We may also find an adjoint linear representation for dKP. This can be derived from the adjoint linear representation of the Toda lattice in the way that we did above. Alternatively, we can define the adjoint directly for the discrete representation. We will see that in taking this second approach one has to consider the linear equations written in an appropriate form before taking the adjoint.

Note first that any solution θ of (3.1) satisfies

$$(\theta^{-1})_{i'j'} = \frac{(\theta\tau)_{i'}(\theta\tau)_{j'}}{(\theta\tau)(\theta\tau)_{i'j'}}((\theta^{-1})_{j'} - (\theta^{-1})_{i'}). \tag{3.10}$$

For the continuous variable Darboux transformation the reciprocal of the solution generating the transformation is, for many classes of problem, a solution of the adjoint transformed equation. We would like this to be the case here and use this to deduce the form of the adjoint system.

Adjoint operators will be defined using the formal inner product for functions a, b of discrete variables n_1, n_2, n_3,

$$\langle a, b\rangle = \sum_{n_1, n_2, n_2 \in \mathbb{Z}} a(n_1, n_2, n_3) b(n_1, n_2, n_3). \tag{3.11}$$

For example,

$$\begin{aligned}\langle a, ub_1\rangle &= \sum_{n_1, n_2, n_2 \in \mathbb{Z}} a(n_1, n_2, n_3) u(n_1, n_2, n_3) b(n_1 + 1, n_2, n_3) \\ &= \sum_{m_1, n_2, n_2 \in \mathbb{Z}} u(n_1 - 1, n_2, n_3) a(n_1 - 1, n_2, n_3) b(n_1, n_2, n_3) \\ &= \langle u_{1'} a_{1'}, b\rangle\end{aligned}$$

and so $(u(\cdot)_1)^\dagger = u_{1'}(\cdot)_{1'}$. If one defines operators $\mathcal{L}^{ij}$ from (3.1) by

$$\mathcal{L}^{ij}(\tau)\phi := \frac{\tau_{ij}\tau}{\tau_i \tau_j}\phi_{ij} + \phi_i - \phi_j = 0 \tag{3.12}$$

then the corresponding adjoint system is

$$\mathcal{L}^{ij}(\tau)^\dagger \psi := \frac{\tau_{i'j'}\tau}{\tau_{i'}\tau_{j'}}\psi_{i'j'} + \psi_{i'} - \psi_{j'} = 0. \tag{3.13}$$

The motivation for the choice of $\mathcal{L}^{ij}$ was given by (3.10) which we now see has the required form

$$\mathcal{L}^{ij}(\theta\tau)^{\dagger}\theta^{-1} = 0. \tag{3.14}$$

Without this motivation, it would be natural to take the adjoint of (3.1) as it stands. The resulting adjoint system is not compatible. Hence it is vital to write (3.1) in the form (3.12) before taking the adjoint.

Hence we take the adjoint linear representation to be

$$\psi_{i'j'} = \frac{\tau_{i'}\tau_{j'}}{\tau\tau_{i'j'}}(\psi_{j'} - \psi_{i'}) \qquad (1 \leqslant i < j \leqslant 3) \tag{3.15}$$

which are compatible if and only if (3.2) is satisfied. Here, given any pair of solutions ρ, ψ of (3.15) we may unambiguously define

$$C(\theta, \phi) = \rho\psi_i - \rho_i\psi \qquad (1 \leqslant i \leqslant 3) \tag{3.16}$$

and then the adjoint Darboux transformation is as follows.

Proposition 3.3. Given any solution ρ of (3.15),

$$\mathrm{aDT}^{\rho}\colon \psi \to \frac{C(\rho, \psi)}{\rho} \qquad \tau \to \rho\tau \tag{3.17}$$

leaves (3.15) invariant.

The N-fold adjoint Darboux transformation is expressed in terms of the Casoratian

$$C(\rho^1, \ldots, \rho^N) = |\boldsymbol{\rho}(0), \boldsymbol{\rho}(1), \ldots, \boldsymbol{\rho}(N-1)|$$

where $\boldsymbol{\rho} = (\rho^1, \ldots, \rho^N)^t$ and $\boldsymbol{\rho}(n) = \boldsymbol{\rho}|_{n_i \to n_i + n}$, the same $i = 1, 2$ or 3 be taken in all columns.

Proposition 3.4. Given N solutions $\rho^1, \ldots, \rho^N$ of (3.15) such that $C(\rho^1, \ldots, \rho^N) \neq 0$,

$$\phi \to \frac{C(\rho^1, \ldots, \rho^N, \phi)}{C(\rho^1, \ldots, \rho^N)} \qquad \tau \to C'(\rho^1, \ldots, \rho^N)\tau \tag{3.18}$$

leaves (3.15) invariant.

Solutions obtained by means of these Darboux transformations will be given in section 5.

4. Binary Darboux transformations

Inversion of the above Darboux transformations and then construction of the binary Darboux transformations relies on the existence of a discrete potential, i.e. a quantity ω defined (up to an additive constant) by difference equations

$$\Delta_i\omega = \alpha^i \qquad (i = 1, 2, 3) \tag{4.1}$$

in which $\Delta_i = (\cdot)_i - (\cdot)$ is the forward-difference operator in discrete variable n_i and the quantities α^i satisfy the compatibility conditions $\Delta_j\alpha^i = \Delta_i\alpha^j$, for $i < j$.

The discrete potential to be used here is given below.

Proposition 4.1. Let $\mathcal{L}^{ij}(\tau)\phi = \mathcal{L}^{ij}(\tau)^{\dagger}\psi = 0$ for some τ. Then there exists a discrete potential $\omega = \omega(\phi, \psi)$ satisfying

$$\Delta_i\omega = \psi\phi_i \qquad (i = 1, 2, 3). \tag{4.2}$$

Now suppose that $\tilde{\phi} = C'(\theta, \phi)/\theta$ is the Darboux transform of ϕ using θ so that

$$\mathcal{L}^{ij}(\theta\tau)\tilde{\phi} = 0 \tag{4.3}$$

and we have seen, in (3.14), that

$$\mathcal{L}^{ij}(\theta\tau)^{\dagger}\theta^{-1} = 0. \tag{4.4}$$

It follows that

$$\Delta_i \theta^{-1}\phi = \theta^{-1}\tilde{\phi}_i. \tag{4.5}$$

By proposition 4.1, using (4.3) and (4.4), we get

$$\phi = \theta\omega(\tilde{\phi}, \theta^{-1}). \tag{4.6}$$

This generalizes to give the inverse Darboux transformation and, in a similar way, the inverse adjoint Darboux transformation.

Proposition 4.2. Given any solution ρ of (3.15),

$$\mathrm{iDT}^{\rho}\colon \phi \to \frac{\omega(\phi, \rho)}{\rho} \qquad \tau \to \rho\tau \tag{4.7}$$

leaves (3.1) invariant.

Proposition 4.3. Given any solution θ of (3.1),

$$\mathrm{iaDT}^{\theta}\colon \psi \to \frac{\omega(\theta, \psi)}{\theta} \qquad \tau \to \theta\tau \tag{4.8}$$

leaves (3.15) invariant.

Note that

$$(\mathrm{DT}^{\theta} \circ \mathrm{iDT}^{(\theta^{-1})})\phi = \phi \qquad (\mathrm{iDT}^{(\theta^{-1})} \circ \mathrm{DT}^{\theta})\phi = \phi + \text{constant}$$

and

$$(\mathrm{aDT}^{\rho} \circ \mathrm{iaDT}^{(\rho^{-1})})\psi = \psi \qquad (\mathrm{iaDT}^{(\rho^{-1})} \circ \mathrm{aDT}^{\rho})\psi = \psi + \text{constant}.$$

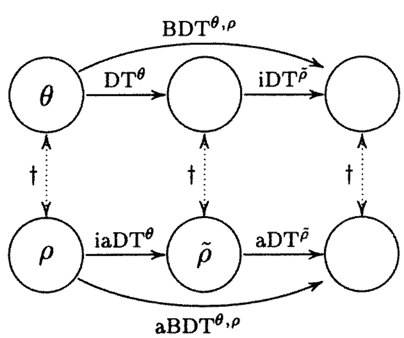
Binary Darboux transformations for the linear system and its adjoint are obtained by composing the above-defined Darboux transformations: the binary Darboux transformation

$$\mathrm{BDT}^{\theta,\rho} := \mathrm{iDT}^{\tilde{\rho}} \circ \mathrm{DT}^{\theta}$$

and the binary adjoint Darboux transformation

$$\mathrm{aBDT}^{\theta,\rho} := \mathrm{aDT}^{\tilde{\rho}} \circ \mathrm{iaDT}^{\theta}$$

where $\tilde{\rho} = \mathrm{iaDT}^{\theta}\,\rho$. This construction is illustrated in the following diagram.



Next we determine the explicit form of the binary Darboux transformations. First, we have

$$\tilde{\rho} = \frac{\omega(\theta, \rho)}{\theta} =: \theta^{-1}\omega^0$$

and so

$$\hat{\phi} = \mathrm{BDT}^{\theta,\rho}\,\phi = \frac{\omega(\theta^{-1}C'(\theta, \phi), \theta^{-1}\omega^0)}{\theta^{-1}\omega^0}.$$

Clearing fractions and taking the n_i-difference of both sides we get

$$\Delta_i(\theta^{-1}\omega^0\hat{\phi}) = \omega^0\Delta_i(\theta^{-1}\phi) = \Delta_i(\theta^{-1}\omega^0\phi) - (\Delta_i\omega^0)\theta_i^{-1}\phi_i.$$

Here we use a difference version of the Liebniz rule

$$\Delta_i(ab) = a(\Delta_i b) + (\Delta_i a)b_i.$$

So finally we get

$$\Delta_i(\theta^{-1}\omega^0\hat{\phi}) = \Delta_i(\theta^{-1}\omega^0\phi) - \rho\phi_i$$

and hence

$$\hat{\phi} = \phi - \theta\,\frac{\omega(\phi, \rho)}{\omega(\theta, \rho)}.$$

A similar, much simpler, calculation may be performed for the binary adjoint transformation and then we have the following results.

Proposition 4.4. Let θ and ρ satisfy (3.1) and (3.15) respectively then

$$\mathrm{BDT}^{\theta,\rho}: \phi \to \phi - \theta\frac{\omega(\phi, \rho)}{\omega(\theta, \rho)} \qquad \tau \to \omega(\theta, \rho)\tau \tag{4.9}$$

$$\mathrm{aBDT}^{\theta,\rho}: \psi \to \psi - \rho\frac{\omega(\theta, \psi)}{\omega(\theta, \rho)} \qquad \tau \to \omega(\theta, \rho)\tau \tag{4.10}$$

leave (3.1) and (3.15) respectively invariant.

These binary Darboux transformations may be iterated and the formulae describing the N-fold transformations are given below.

Proposition 4.5. Let $\boldsymbol{\theta} = (\theta^1, \ldots, \theta^N)^t$ and $\boldsymbol{\rho} = (\rho^1, \ldots, \rho^N)^t$ satisfy (3.1) and (3.15) respectively then

$$\phi \to \frac{\begin{vmatrix} \phi & \omega(\phi, \boldsymbol{\rho}^t) \\ \boldsymbol{\theta} & \omega(\boldsymbol{\theta}, \boldsymbol{\rho}^t) \end{vmatrix}}{|\omega(\boldsymbol{\theta}, \boldsymbol{\rho}^t)|} \qquad \tau \to |\omega(\boldsymbol{\theta}, \boldsymbol{\rho}^t)|\tau \tag{4.11}$$

$$\psi \to \frac{\begin{vmatrix} & \psi\boldsymbol{\rho}^t \\ \omega(\boldsymbol{\theta}, \psi) & \omega(\boldsymbol{\theta}, \boldsymbol{\rho}^t) \end{vmatrix}}{|\omega(\boldsymbol{\theta}, \boldsymbol{\rho}^t)|} \qquad \tau \to |\omega(\boldsymbol{\theta}, \boldsymbol{\rho}^t)|\tau \tag{4.12}$$

leave (3.1) and (3.15) respectively invariant.

5. Explicit solutions obtained by Darboux transformations

Finally, we present explicit examples of the classes of solutions that may be obtained by means of the Darboux transformations derived above. In its canonical form (2.4), the vacuum solution of the dKP equation is $\tau = \tau_0 = 1$. Thus, by virtue of the transformation (2.5), the natural choice for the vacuum in the version of the equation (2.6) that we use is

$$\tau_0 = \prod_{i<j=1}^{3} \left(\frac{a_i - a_j}{a_i a_j}\right)^{n_i n_j}. \tag{5.1}$$

With this choice, the linear problem (2.7) reads

$$\phi_{ij} = \left(\frac{a_i a_j}{a_i - a_j}\right)(\phi_j - \phi_i) \qquad (i < j) \tag{5.2}$$

and the basic eigenfunctions, depending on a single parameter p are found to be

$$\phi(p) = \prod_{i=1}^{3} \left(\frac{a_i}{a_i p - 1}\right)^{n_i}. \tag{5.3}$$

In a similar way the basic adjoint eigenfunctions, depending on parameter q, are

$$\psi(q) = \prod_{i=1}^{3} \left(\frac{a_i}{a_i q - 1}\right)^{-n_i} \tag{5.4}$$

and for these eigenfunction we may integrate (4.1) to obtain the potential

$$\omega(\phi, \psi) = c + \frac{pq}{q - p} \prod_{i=1}^{3} \left(\frac{a_i q - 1}{a_i p - 1}\right)^{n_i} \tag{5.5}$$

where c is a constant.

Given the above expression it is straightforward to write down the following explicit solutions for (2.6)

$$\tau = C'(\phi_1, \ldots, \phi_N)\tau_0 \tag{5.6}$$

where $\phi_i = \phi(p_i) + \alpha_i \phi(p_i')$ where $\phi(p)$ is given by (5.3) and p_i, p_i' and α_i are arbitrary constants;

$$\tau = C(\psi_1, \ldots, \psi_N)\tau_0 \tag{5.7}$$

where $\psi_i = \psi(q_i) + \beta_i \psi(q_i')$ where $\psi(q)$ is given by (5.4) and q_i, q_i' and β_i are arbitrary constants;

$$\tau = \det(\omega_{i,j})\tau_0 \tag{5.8}$$

where $\omega_{i,j}$ is given by (5.5) with $p = p_i$ and $q = q_j$ and $c = c_{ij}$. These solutions are all soliton-like in the sense that a field such as

$$U = \frac{\tau_{ij}\tau}{\tau_i \tau_j} \tag{5.9}$$

contains coherent structures interacting elastically and also, in a continuum limit, each tends to a soliton solution.

Further, by regarding the eigenfunctions $\phi(p)$ as a generating function in p we obtain a sequence of (essentially) polynomial expressions which satisfy the linear equations (5.2) and may be regarded as discrete analogues of the Schur polynomials. The resulting solutions of dKP are discrete analogues of the Schur function solutions of the KP hierarchy.

We define polynomials $h^{(k)}(n_1, n_2, n_3)$ by

$$\phi(p) = \prod_{j=1}^{3} (-a_j)^{n_j} \sum_{k=0}^{\infty} h^{(k)} p^k \tag{5.10}$$

so that, for example,

$$h^{(0)} = 1 \qquad h^{(1)} = \sum_{i=1}^{3} a_i n_i \qquad \text{and} \qquad h^{(2)} = \tfrac{1}{2}\left(\sum_{i=1}^{3} a_i n_i\right)^2 + \tfrac{1}{2}\sum_{i=1}^{3} a_i^2 n_i$$

and in general the $h^{(k)}$ satisfy

$$(a_i - a_j) h_{ij}^{(k)} = a_i h_i^{(k)} - a_j h_j^{(k)}.$$

In this way we obtain Casoratian-type polynomial solutions of the dKP equation. Using an invariance of the form $\tau \to \prod_{i=1}^{3} \alpha_i^{n_i} \tau$ to remove an irrelevant multiplicative factor, these may be written in their simplest form as

$$\tau = \mathcal{C}'(h^{(i_1)}, \ldots, h^{(i_n)})\tau_0$$

for nonnegative integers $i_1 < i_2 < i_3 < \ldots < i_n$.

Alternative representations of these solutions may also be found in the forms given in (5.7) and (5.8).

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Appendix. Some proofs

This section contains proofs of some of the propositions in the main text. One each of the basic and binary Darboux transformations is proved, the omitted proofs are very similar.

A.1. Proof of (3.2)

Let

$$F := C'(\theta^1, \ldots, \theta^N) = |\boldsymbol{\theta}(0), \ldots, \boldsymbol{\theta}(1-N)|$$
$$G := C'(\theta^1, \ldots, \theta^N, \phi) = |\boldsymbol{\theta}^+(0), \ldots, \boldsymbol{\theta}^+(-N)|$$

where $\boldsymbol{\theta}^+ = (\theta^1, \ldots, \theta^N, \phi)^t$. To verify that (3.1) is invariant under (3.9) we must show that

$$\frac{\tau\tau_{ij}}{\tau_i\tau_j} F G_{ij} + G_i F_j - G_j F_i = 0 \qquad (i < j). \tag{A1}$$

The basic property we use in proving this is the following formula, deduced from (3.3): for $n < 0$,

$$\boldsymbol{\theta}_i(n) = \boldsymbol{\theta}(n+1) + \sum_{k=0}^{n+1} \alpha^k \boldsymbol{\theta}_i(k)$$

where α^k are scalars. It follows that

$$F_i = |\boldsymbol{\theta}_i, \boldsymbol{\theta}(0), \ldots, \boldsymbol{\theta}(2-N)|$$
$$G_i = |\boldsymbol{\theta}_i^+, \boldsymbol{\theta}^+(0), \ldots, \boldsymbol{\theta}^+(1-N)|$$
$$G_{ij} = -\frac{\tau\tau_{ij}}{\tau_i\tau_j}|\boldsymbol{\theta}_i^+, \boldsymbol{\theta}_j^+, \boldsymbol{\theta}^+(0), \ldots, \boldsymbol{\theta}^+(2-N)|.$$

Substituting into the left-hand side of (A1) gives

$$\begin{aligned}&|\boldsymbol{\theta}_i^+, \boldsymbol{\theta}_j^+, \boldsymbol{\theta}^+(0), \ldots, \boldsymbol{\theta}^+(2-N)||\boldsymbol{\theta}(0), \ldots, \boldsymbol{\theta}(1-N)|\\ &\qquad -|\boldsymbol{\theta}_i^+, \boldsymbol{\theta}^+(0), \ldots, \boldsymbol{\theta}^+(1-N)||\boldsymbol{\theta}_j, \boldsymbol{\theta}(0), \ldots, \boldsymbol{\theta}(2-N)|\\ &\qquad +|\boldsymbol{\theta}_j^+, \boldsymbol{\theta}^+(0), \ldots, \boldsymbol{\theta}^+(1-N)||\boldsymbol{\theta}_i, \boldsymbol{\theta}(0), \ldots, \boldsymbol{\theta}(2-N)|\end{aligned}$$

which is equal to zero as it is the expansion of the vanishing determinant

$$\begin{vmatrix} \boldsymbol{\theta}_i^+ & \boldsymbol{\theta}_j^+ & \boldsymbol{\theta}^+(0) & \cdots & \boldsymbol{\theta}^+(2-N) & 0 & \cdots & 0 & \boldsymbol{\theta}^+(1-N) \\ \boldsymbol{\theta}_i & \boldsymbol{\theta}_j & 0 & \cdots & 0 & \boldsymbol{\theta}(0) & \cdots & \boldsymbol{\theta}(2-N) & \boldsymbol{\theta}(1-N) \end{vmatrix}.$$

A.2. Proof of (4.5)

We use the following notation: $\boldsymbol{\omega} = \omega(\phi, \boldsymbol{\rho})$ is an N-vector and $\Omega = \omega(\boldsymbol{\theta}, \boldsymbol{\rho}^t)$ an $N \times N$ matrix. Let

$$F = |\Omega| \qquad G = \begin{vmatrix} \phi & \boldsymbol{\omega}^t \\ \boldsymbol{\theta} & \Omega \end{vmatrix}.$$

Again we need to show that (A1) is satisfied. Now

$$F_i = |\boldsymbol{\theta}_i\boldsymbol{\rho}^t + \Omega| = F - \begin{vmatrix} 0 & \boldsymbol{\rho}^t \\ \boldsymbol{\theta}_i & \Omega \end{vmatrix}$$
$$G_i = \begin{vmatrix} \phi_i & \boldsymbol{\omega}^t \\ \boldsymbol{\theta}_i & \Omega \end{vmatrix}$$
$$G_{ij} = \frac{\tau_i\tau_j}{\tau_{ij}\tau}\left[G_j - G_i + \begin{vmatrix} 0 & 0 & \boldsymbol{\rho}^t \\ \phi_j & \phi_i & \boldsymbol{\omega}^t \\ \boldsymbol{\theta}_j & \boldsymbol{\theta}_i & \Omega \end{vmatrix}\right].$$

Then the left-hand side of (A1) becomes

$$\begin{vmatrix} 0 & 0 & \boldsymbol{\rho}^t \\ \phi_j & \phi_i & \boldsymbol{\omega}^t \\ \boldsymbol{\theta}_j & \boldsymbol{\theta}_i & \Omega \end{vmatrix}|\Omega| - \begin{vmatrix} \phi_i & \boldsymbol{\omega}^t \\ \boldsymbol{\theta}_i & \Omega \end{vmatrix}\begin{vmatrix} 0 & \boldsymbol{\rho}^t \\ \boldsymbol{\theta}_j & \Omega \end{vmatrix} + \begin{vmatrix} \phi_j & \boldsymbol{\omega}^t \\ \boldsymbol{\theta}_i & \Omega \end{vmatrix}\begin{vmatrix} 0 & \boldsymbol{\rho}^t \\ \boldsymbol{\theta}_i & \Omega \end{vmatrix}$$

which vanishes because of a Jacobi identity.

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